CMPT 409/981: Quantum Circuits and Compilation
Assignment 1

Due October 7th at the start of class
on paper or by email to the instructor

Question 1 [2 points]: GHZ states

Show that the 3-qubit state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

cannot be written as a tensor product of 1-qubit states,

$$|\psi\rangle \otimes |\varphi\rangle \otimes |\theta\rangle.$$

Question 2 [3 points]: Entangling operations

Show that the matrix

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is an entangling operation by giving an explicit 2-qubit product state $|\psi\rangle \otimes |\phi\rangle$ and showing that the state $CZ(|\psi\rangle \otimes |\phi\rangle)$ is entangled.

Question 3 [7 points]: Strange occurrences

Recall that

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

is an example of a Bell state. There are 4 Bell states in total, listed below:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Together, these states give an orthonormal basis $\mathcal{B} = \{\Phi^+, \Phi^-, \Psi^+, \Psi^-\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$ called the *Bell basis*. As entangled states they can be used to perform *spooky action at a distance*, some effects of which we will explore in this question.

1. Show that for any 2x2 matrix A, $A \otimes I|\Phi^+\rangle = I \otimes A^t|\Phi^+\rangle$, where A^t is the transpose of A.

2. Since $\mathcal{B}$ is an orthonormal basis, we can measure a 2-qubit state with respect to this basis, using $\{|v\rangle\langle v| : |v\rangle \in \mathcal{B}\}$ as our measurement operators. Given a 3 qubit system prepared in the state $|\psi\rangle \otimes |\Phi^+\rangle$ where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, show that the final state after measuring the FIRST TWO qubits with respect to the Bell basis and obtaining measurement result $|\Phi^+\rangle$ is

$$|\Phi^+\rangle \otimes |\psi\rangle$$

3. Show that if in the prior question we instead prepared the initial state as

$$|\psi\rangle \otimes (A \otimes I|\Phi^+\rangle)$$

then the state after obtaining measurement outcome $|\Phi^+\rangle$ is

$$|\Phi^+\rangle \otimes (A^t|\psi\rangle)$$

Hint: there is a simple algebraic proof that doesn't require explicit matrix calculations. To find this proof, you may find it helpful to ignore the post-measurement normalization — that is, given measurement operators $\{M_m\}$ and state $|\varphi\rangle$, the final unnormalized state after receiving measurement result m is $M_m|\varphi\rangle$. Often working "up to normalization" simplifies calculations significantly.

4. Recall that

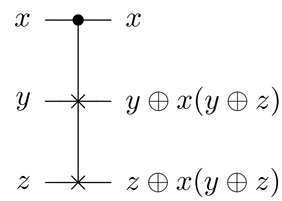
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Design a circuit using H and $CNOT$ gates that will map Bell states to states of the computational basis, so that we may implement Bell basis measurements via computational basis measurements. It may be helpful to use the following characterization ($x, y \in \{0, 1\}$):

$$CNOT : |x\rangle|y\rangle \mapsto |x\rangle|x \oplus y\rangle$$
$$H : |x\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle)$$

Question 4 [2 points]: Another universal gate?

The FREDKIN gate is a 3-bit reversible gate implementing which swaps the last two bits if and only if the first bit is 0. Graphically, the FREDKIN gate can be written as



Show that the FREDKIN gate, together with ancillas initialized in the 0 or 1 state is universal for reversible computation. You may use the fact that the TOFFOLI gate is universal (with ancillas in the 0 or 1 state).

Question 5 [4 points]: Majority rules

The *majority* function $maj : \{0,1\}^3 \to \{0,1\}$ computes the majority value of 3 bits — that is, if 2 or more of x, y, z are 1, then $maj(x, y, z) = 1$, otherwise $maj(x, y, z) = 0$.

1. Verify that $maj(x, y, z) = xy \oplus xz \oplus yz$

   Hint: the function is symmetric in all of its inputs, so you only need to consider four cases

2. Give a reversible circuit implementing the majority function using 3 Toffoli gates and 1 ancilla initialized in the 0 state.

3. Give an alternate factorization of the maj function that uses only 2 multiplications and use this decomposition to give a reversible circuit implementing maj with only 2 Toffoli gates and arbitrarily many CNOT gates

4. **(Bonus 2pts)**: Give an implementation of the maj function using only 1 Toffoli gate (you can use as many CNOT gates and ancillas as you like)

Question 6 [10 points]: Diagonalization

We say that two matrices A, B are *simultaneously diagonalizable* if there exists some unitary U such that $A = U\Lambda_A U^\dagger$, $B = U\Lambda_B U^\dagger$. It can be shown that two Hermitian operators A, B are simultaneously diagonalizable if and only if they commute, i.e.,

$$AB = BA$$

Moreover, if A and B commute, then it can be shown that $e^{A+B} = e^A e^B$.

In this question we use simultaneous diagonalization to *efficiently* implement a circuit for simulating a *Hamiltonian* of the form

$$e^{i(\theta_1 X\otimes Z + \theta_2 Z\otimes X)}$$

The design of such circuits is the primary compilation question for quantum algorithms simulating quantum systems.

1. As a warmup, diagonalize $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ by finding orthonormal vectors $|+_Y\rangle$, $|-_Y\rangle$ such that

$$\begin{aligned} Y|+_Y\rangle &= |+_Y\rangle \\ Y|-_Y\rangle &= -|-_Y\rangle \end{aligned}$$

2. Show that $X \otimes Z$ and $Z \otimes X$ commute. (**Hint:** recall that $XZ = -ZX$)

3. Find an orthonormal basis for the 4-dimensional *joint eigenspace* of $X \otimes Z$ and $Z \otimes X$. That is, find some set $\{|v_i\rangle\}$ of 4 unit vectors such that

$$\begin{aligned} (X \otimes Z)|v_i\rangle &= \lambda_{a,i}|v_i\rangle \\ (Z \otimes X)|v_i\rangle &= \lambda_{b,i}|v_i\rangle \\ \forall i \neq j, \qquad \langle v_i|v_j\rangle &= 0 \end{aligned}$$

4. Use these eigenvectors to design a unitary U simultaneously diagonalizing $X \otimes Z$ and $Z \otimes X$ as $Z \otimes I$ and $I \otimes Z$, respectively.

5. Show that

$$e^{i(\theta_1 X \otimes Z + \theta_2 Z \otimes X)} = U \left(e^{i\theta_1 Z} \otimes e^{i\theta_2 Z} \right) U^\dagger$$